Fibonacci numbers

The **Fibonacci** sequence is named after Italian mathematician Leonardo of Pisa, known as Fibonacci:

https://en.wikipedia.org/wiki/Fibonacci_number

The **Fibonacci** numbers f_n = f(n) are the numbers characterized by the fact that every number after the first two is the sum of the two preceding ones. They are defined with the next recurrent relation:

$$f(n) = \begin{cases} 0, if \ n = 0 \\ 1, if \ n = 1 \\ f(n-1) + f(n-2) \end{cases}$$

So $f_0 = 0$, $f_1 = 1$, $f_n = f_{n-1} + f_{n-2}$.

The Fibonacci sequence has the form

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, …

Example. Fill integer array *fib* with Fibonacci numbers (fib[i] = f_i):

```
#include <stdio.h>

int i, n, fib[47];

int main(void)
{
  scanf("%d",&n);

  fib[0] = 0; fib[1] = 1;
  for(i = 2; i <= n; i++)
    fib[i] = fib[i-1] + fib[i-2];

  printf("%d\n",fib[n]);
  return 0;
}
```

i	0	1	2	3	4	5	6	7	8	9	10	...
fib[i]	0	1	1	2	3	5	8	13	21	34	55	...

The biggest Fibonacci number that fits into `int` type is

f_{46} = 1836311903

The biggest Fibonacci number that fits into `long long` type is

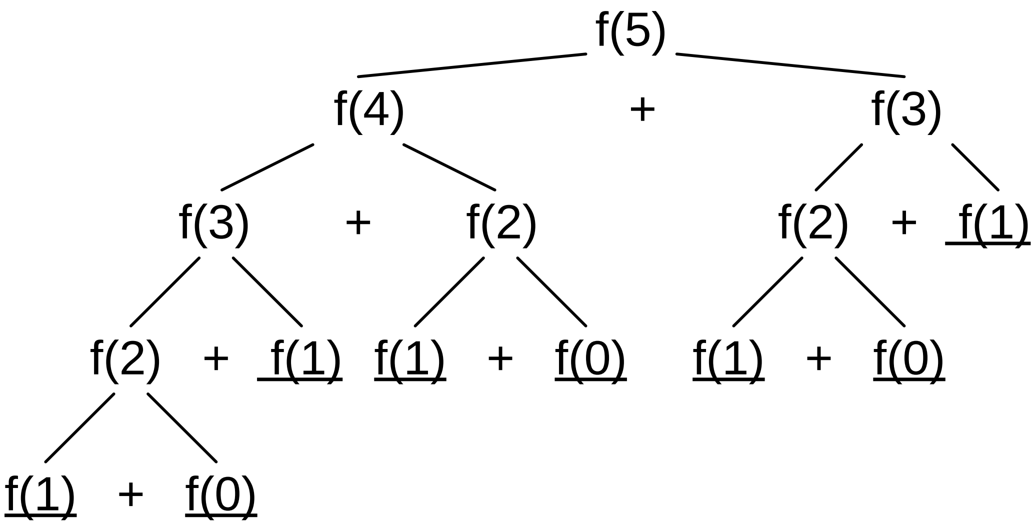
f_{92} = 7540113804746346429

If you want to find Fibonacci number f_n for n > 92, use **BigInteger** type.

Example. Find f(n) – the n-th Fibonacci number with recursion:

```
#include <stdio.h>
```

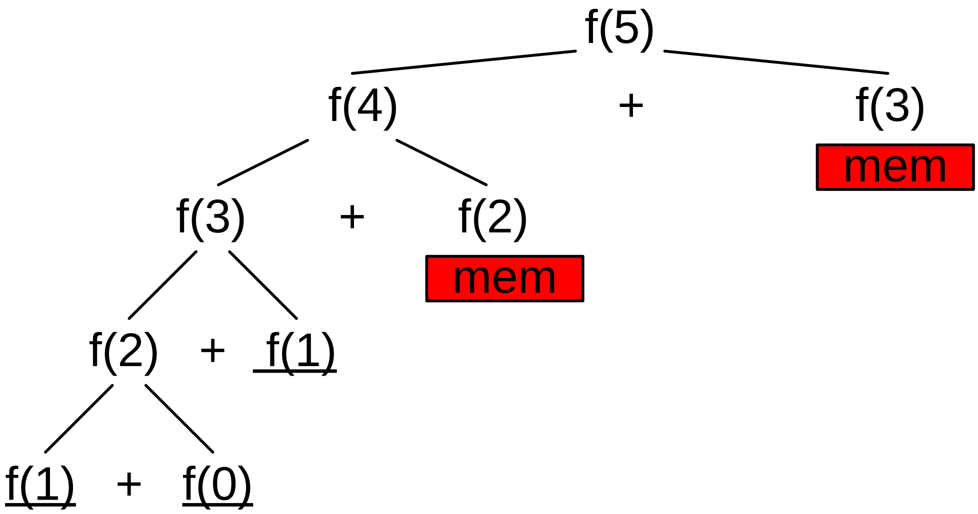
```c
int n;

int fib(int n)
{
  if (n == 0) return 0;
  if (n == 1) return 1;
  return fib(n-1) + fib(n - 2);
}

int main(void)
{
  scanf("%d",&n);
  printf("%d\n",fib(n));
  return 0;
}
```



Example. Find f(*n*) – the *n*-th Fibonacci number with recursion + memoization:

```c
#include <stdio.h>
#include <string.h>

int n, fib[46];

int f(int n)
{
  // base case
  if (n == 0) return 0;
  if (n == 1) return 1;

  // if the value fib[n] is ALREADY found, just return it
  if (fib[n] != -1) return fib[n];

  // if the value fib[n] is not found, calculate and memoize it
  return fib[n] = f(n-1) + f(n - 2);
}

int main(void)
{
  scanf("%d",&n);
```

```
  // fib[i] = -1 means that this value is not calculated yet
  memset(fib,-1,sizeof(fib));

  printf("%d\n",f(n));
  return 0;
}
```



Java code

```java
import java.util.*;

public class Main
{
  static int fib[] = new int[46];

  static int f(int n)
  {
    if (n == 0) return 0;
    if (n == 1) return 1;
    if (fib[n] != -1) return fib[n];
    return fib[n] = f(n-1) + f(n - 2);
  }

  public static void main(String[] args)
  {
    Scanner con = new Scanner(System.in);
    int n = con.nextInt();
    Arrays.fill(fib, -1);
    System.out.println(f(n));
    con.close();
  }
}
```

E-OLYMP [4730. Fibonacci] Fibonacci numbers is a sequence of numbers F(n), given by the formula:

$$F(0) = 1, F(1) = 1, F(n) = F(n-1) + F(n-2)$$

Given value of n ($n \le 45$). Find the n-th Fibonacci number.

► Implement a recursive function with memoization.

NO two one’s in a row

Find the number of sequences of length *n*, consisting only of zeros and ones, that do not have two one’s in a row.

Let *f*(*n*) be the number of sequences consisting of 0 and 1 of length *n* that do not have two one’s in a row.

0
1

f(1) = 2

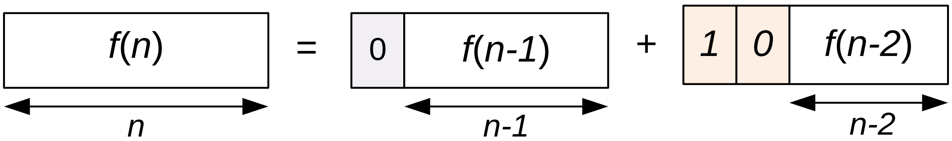
0	0
0	1
1	0

f(2) = 3

0	0	0
0	0	1
0	1	0
1	0	0
1	0	1

f(3) = 5

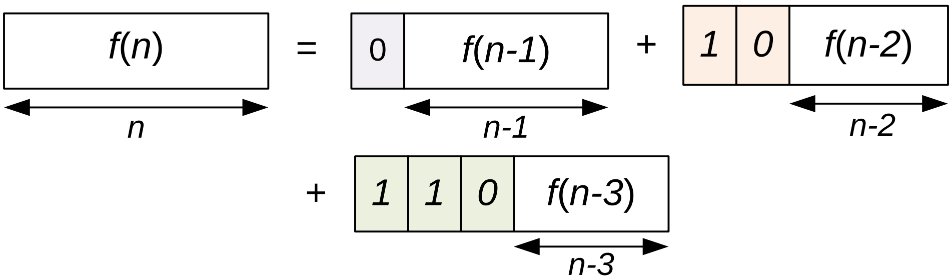
If the first number in the sequence is 0, then starting from the second place we can build *f*(*n* – 1) sequences. If the first number in the sequence is 1, then second number should be 0.



We have Fibonacci numbers with base cases f(1) = 2, f(2) = 3.

E-OLYMP [263. Three ones](#) Find the number of sequences of length *n*, consisting only of zeros and ones, that do not have three one’s in a row.

► Let *f*(*n*) be the number of required sequences consisting of 0 and 1 of length *n*. If the first number in the sequence is 0, then starting from the second place we can build *f*(*n* – 1) sequences. If the first number in the sequence is 1, then second number can be any (0 or 1). If second number is 0, on the next *n* – 2 free places we can construct *f*(*n* – 2) sequences. If second number is 1, the third number must be exactly 0, and starting from the forth place we can construct *f*(*n* – 3) sequences.



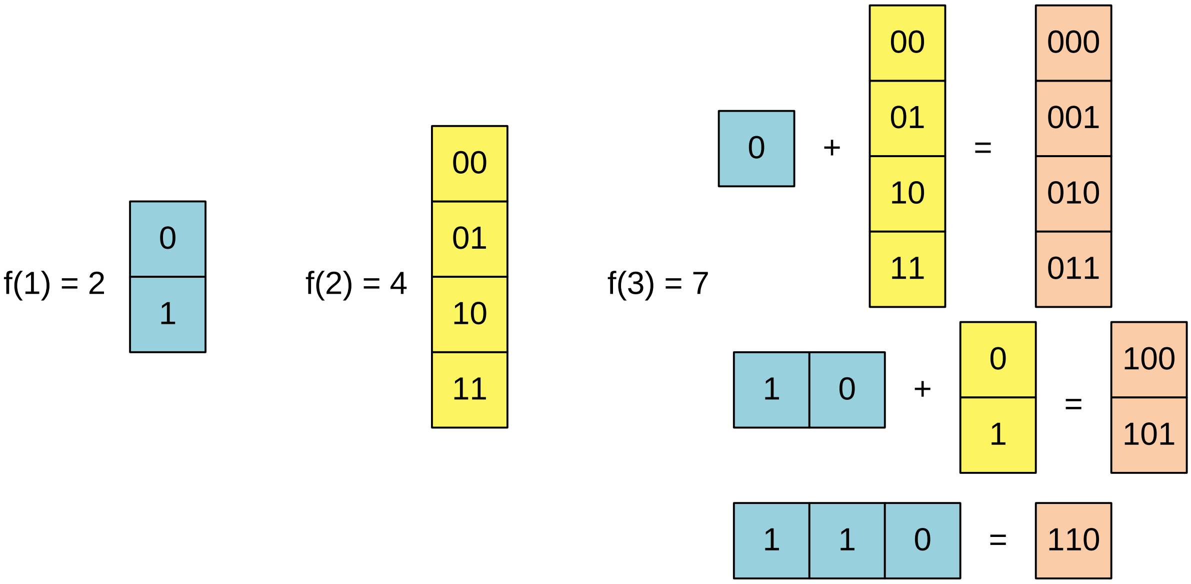
We have the recurrence: f(*n*) = f(*n* – 1) + f(*n* – 2) + f(*n* – 3). Now we must calculate the initial values:

f(1) = 2, since there are two sequence of lengths 1: 0 and 1.

f(2) = 4, since there are four sequence of lengths 2: 00, 01, 10 and 11.

f(3) = 7, since there are seven sequence of lengths 3: 000, 001, 010, 011, 100, 101 and 110.

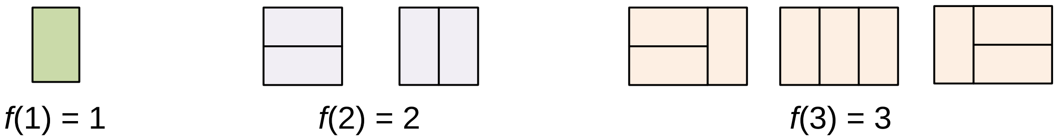
Do not forget to run all operations modulo 12345.



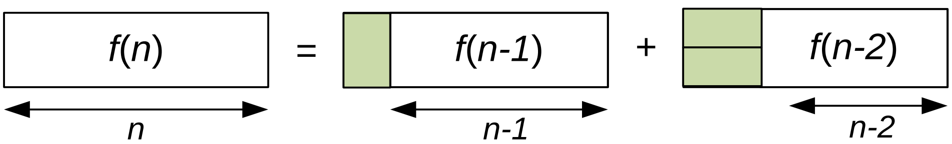
E-OLYMP [4469. Domino](#) Find the number of ways to cover a rectangle $2 \times n$ with domino of size 2×1. The coverings that turn themselves into symmetries are considered different.

► Let f(n) be the number of ways to cover the $2 \times n$ rectangle with 2×1 dominoes. Obviously, that

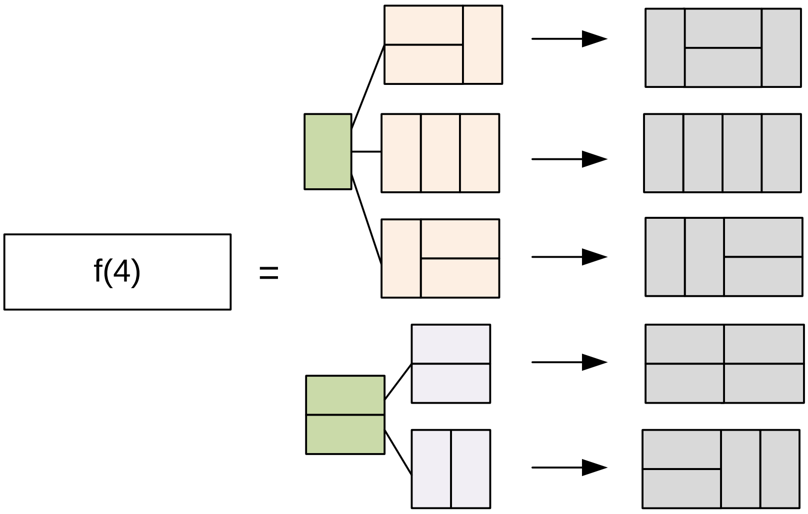
- $f(1) = 1$, one vertical domino;
- $f(2) = 2$, two vertical or two horizontal dominoes.



Consider an algorithm for computing $f(n)$. You can put one domino vertically and then cover a rectangle of length $n - 1$ in $f(n - 1)$ ways, or put two dominoes horizontally and then cover a rectangle of length $n - 2$ in $f(n - 2)$ ways. That is, $f(n) = f(n - 1) + f(n - 2)$.



So $f(n)$ is the Fibonacci number.



Since n < 65536, long arithmetic or Java programming language should be used.

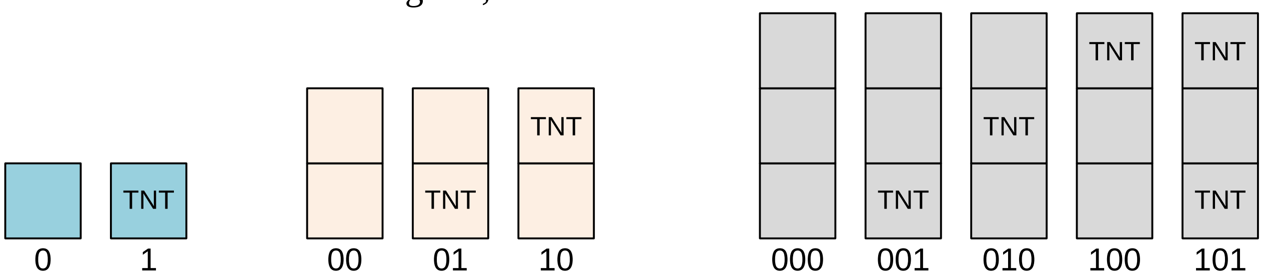
E-OLYMP [**5091. Explosive containers**]() You have two types of boxes: with trotyl (TNT) or without. You must build with boxes a tower of height n. In how many ways can you do it if it is forbidden to put TNT box on TNT box because of explosion.

► Let's code the empty box with 0 and the box with TNT with 1. In the problem we must find the number of strings of length n consisting of 0 and 1, in which two ones are not adjacent. The answer to the problem will be the Fibonacci number f(n):

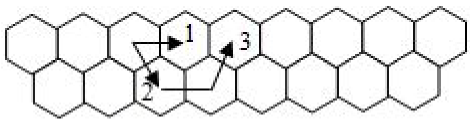
$$f(n) = \begin{cases} 2, if \; n = 1 \\ 3, if \; n = 2 \\ f(n-1) + f(n-2) \end{cases}$$

Consider all possible towers of height n = 1, n = 2, n = 3. Each of them corresponds a sequence of 0 and 1. There are:

- two towers of height 1;
- three towers of height 2;
- five towers of height 3;

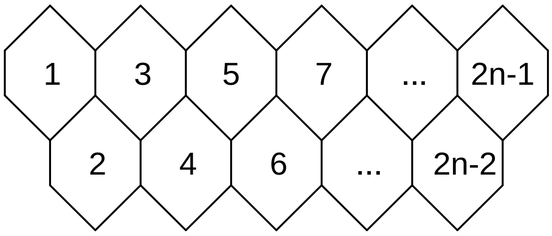


E-OLYMP [**5092. Honeycomb**]() The bee can go in honeycomb as shown in the figure – with moves 1 and 2 from upper row and with move 3 from the lower.

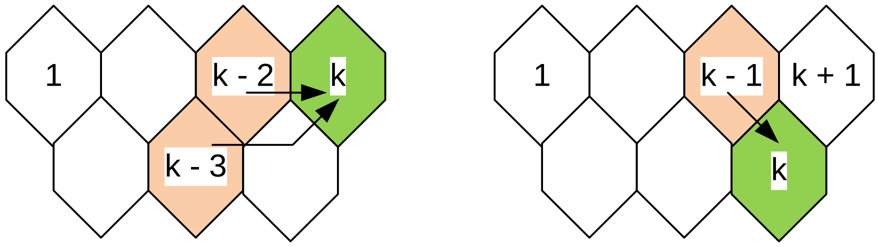


Find the number of ways to get from the first cell of the top row to the last cell of the same row.

► Enumerate the honeycomb in the next way:



Let f(k) be the number of ways to get from the first honeycomb into the k-th one. If upper row contains n honeycomb, the number of rightmost honeycomb of upper row has number $2n - 1$. So the answer to the problem will be f($2n - 1$).



If k-th honeycomb is located in the upper row, the bee can come into it either from ($k - 2$)-th honeycomb, or from ($k - 3$)-th. So $f(k) = f(k - 2) + f(k - 3)$ for odd k.

If k-th honeycomb is located in the lower row, the bee can come into it only from ($k - 1$)-th honeycomb. So $f(k) = f(k - 1)$ for even k.

Calculate the base cases separately: f(1) = 1, f(2) = 1, f(3) = 1.

E-OLYMP [8295. Fibonacci string generation] Generate the n-th Fibonacci string that is defined with the next recurrent formula:

- f(0) = "*a*";
- f(1) = "*b*";
- f(n) = f(n – 1) + f(n – 2), where "+" operation means concatenation

For example, f(3) = f(2) + f(1) = (f(1) + f(0)) + f(1) = "*b*" + "*a*" + "*b*" = "*bab*".

► Implement a recursive function that generates the n-th Fibonacci string.

```
string f(int n)
{
  if (n == 0) return "a";
  if (n == 1) return "b";
  return f(n-1) + f(n-2);
}
```

Read input value of n and print the n-th Fibonacci string.

```
cin >> n;
cout << f(n) << endl;
```